Body symmetry – studies in the Framsticks simulator

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Details of this research are available in [JK06; JK08].

Model components

Framsticks

Reminder of the creature model

Symmetr itself

Static symmetry

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References

Body

- Parts (3D location & orientation)
- Joints
- Brain
 - Neurons (embodied or not)
 - Connections



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Properties

Physical and biological: lengths, sizes, masses, etc.

Model constraints

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- at most one Joint can directly connect two Parts
- each Joint must be connected with two distinct Parts
- all Parts must be directly or indirectly connected with each other

Native simulation engine – *MechaStick*

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- physics-based: create real-world feeling to intuitively understand behaviors
- not necessarily very accurate but fast performance matters demo
- Parts: atomic physical objects
- Joints: description of internal forces and constraints, visualized as sticks

Native simulation engine – *MechaStick*

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- physics-based: create real-world feeling to intuitively understand behaviors
- not necessarily very accurate but fast performance matters demo
- Parts: atomic physical objects
- Joints: description of internal forces and constraints, visualized as sticks

- rigid bodies: no
- volume bodies: no
- collision detection within creatures: no

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Figure: Vitruvian Man

Symmetry. What's that?



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Definition

Symmetry is an intrinsic property of a mathematical object which causes it to remain *invariant* under certain classes of transformations (such as rotation, reflection, inversion, or more abstract operations).

Symmetry in various disciplines



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Figure: The Taj Mahal, Agra, India, 1648 r.

- Physics
- Math
- Music
- Poetry
- Architecture

Symmetry in various disciplines

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Figure: The Taj Mahal, Agra, India, 1648 r.

- Physics
- Math
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- Poetry
- Architecture
- Moral symmetry (tit for tat)

Symmetry



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References

Herman Weyl, "Symmetry"

Symmetry is an idea which has guided man through the centuries to the understanding and the creation of **order**, **beauty** and **perfection**.

Symmetry in biology

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Figure: Symmetry – a popular evolutionary concept.

- Popular evolutionary concept
- Usually bilateral symmetry (the bilateria)
- Oldest known symmetrical organism: Vernanimalcula (600 mln years ago)
- Notable asymmetrical exceptions: sponges.

Sponges

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Figure: Sponges

Symmetry everywhere?



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- Animals are symmetrical only superficially and only in a macro scale
- Asymmetry in chemistry
- Alice's cat
- DNA is clockwise

What is on the other side of looking glass?

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Figure: On the other side

- Is the reflected world possible?
- Let us reflect the whole universe...from stars till atoms...

What is on the other side of looking glass?

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Figure: On the other side

- Is the reflected world possible?
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- A reflection of neutrino is impossible \rightarrow reflected world is impossible. . .

What is on the other side of looking glass?

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Figure: On the other side

- Is the reflected world possible?
- Let us reflect the whole universe...from stars till atoms...
- A reflection of neutrino is impossible \rightarrow reflected world is impossible...
- unless we also reflect the arrow of time...

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An open problem.

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• An open problem.

• Females of some species prefer males with the most symmetrical sexual ornaments.

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- An open problem.
- Females of some species prefer males with the most symmetrical sexual ornaments.
- For humans, there are proved positive correlations between facial symmetry and health and
- between facial symmetry and perception of beauty

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- Intuition: bilateral symmetry resulted from the direction of movement of living creatures
- Proof: positive correlations between locomotive efficiency and morphological symmetry
- If so, why in the world of flowers symmetry (usually radical) is so common? Certainly not for locomotion.

Numerical measure of symmetry - motivations

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Further research

- Common language is capable to express various degrees of symmetry, but no general numerical symmetry definition exists
- Natural, binary notion of symmetry is insufficient for research
- Numerical measure of symmetry could allow determining the extent to what an object is symmetrical, but also...
- if one object is more symmetrical than another.

Numerical measure of symmetry - motivations

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Symmetry is not such a popular concept in artificial worlds, so in order to study the phenomenon of symmetry and its implications, there is a need for defining a **numerical**, fully **automated** and **objective** measure of symmetry for creatures living in artificial environments

More motivations

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Further research

- A tool for researcher (earlier: "similarity" measure)
- Possible research applications:
 - Do symmetrical creatures move faster/further/more reliably?
 - Do symmetrical creatures perform better in environments they were not evolved in?
 - Does evolution produce more symmetrical creatures in worlds with difficult terrain/bigger/smaller gravitation?
 - ...and more

Creature's model (framsticks)

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Only skeleton is taken into account.



Solid 3D objects

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• The Symmetry Condition. If c is perfectly bilaterally symmetrical, then sym(c) = 1.0.

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- The Symmetry Condition. If c is perfectly bilaterally symmetrical, then sym(c) = 1.0.
- The Asymmetry Condition. If c is completely asymmetrical then sym(c) = 0.0.

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- The Symmetry Condition. If c is perfectly bilaterally symmetrical, then sym(c) = 1.0.
- The Asymmetry Condition. If c is completely asymmetrical then sym(c) = 0.0.
- The Common Sense Condition. If c_1 is more symmetrical than c_2 , then $sym(c_1) > sym(c_2)$.

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- The Common Sense Condition. If c_1 is more symmetrical than c_2 , then $sym(c_1) > sym(c_2)$.
- The Proportional Difference Condition. The difference between $sym(c_1)$ and $sym(c_2)$ should correspond to the difference in anatomical symmetry between c_1 and c_2 .

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- The Common Sense Condition. If c_1 is more symmetrical than c_2 , then $sym(c_1) > sym(c_2)$.
- The Proportional Difference Condition. The difference between $sym(c_1)$ and $sym(c_2)$ should correspond to the difference in anatomical symmetry between c_1 and c_2 .
- The Scalability Condition. The proposed measure should be robust against scaling: for creature c_2 that is a scaled version of c_1 (body enlarged or diminished), we expect $sym(c_2) = sym(c_1)$.



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Let us denote symmetry of a creature c about plane p as sym(c, p). We say that "a creature is symmetrical" if it is symmetrical about **any plane**, therefore we are looking for a plane that yields the highest symmetry:

$$sym(c) = \max_{p}(sym(c,p))$$

(1)

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Looking for matching sticks...



How to compute sym(c, p)? (1)


How to compute sym(c, p)? (2)



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How to compute sym(c, p)? (3)

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$$sym(c, p) = \max_{\Pi} \left(rac{\sum_{(s_1, s_2) \in \Pi} w_{s_1 s_2} sim(s_1, s_2)}{\sum_{(s_1, s_2) \in \Pi} w_{s_1 s_2}}
ight)$$

$$w_{s_1s_2}=\left\{egin{array}{cc} \mathit{len}(s_1)+\mathit{len}(s_2) & \mathrm{if} & s_1
eq s_2\ \mathit{len}(s_1) & \mathrm{if} & s_1=s_2 \end{array}
ight.$$

$$sim(s_1, s_2) = \exp \frac{-dist^2(s_1, s_2)}{(\alpha \cdot s_f)^2}$$
(4)

(2)

(3)

where α is a constant, and s_f is a creature scale factor.

Sample landscape



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Figure: In order to find the plane of the highest symmetry, we sample the 3-dimensional (α, β, t) space for each creature stick and then perform a local search to further improve the best found plane.

Illustration of symmetry planes (1)

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Figure: Sample creatures, estimation of their symmetry planes and symmetry values. Values of symmetry are: 1.0, 1.0, 0.99

Illustration of symmetry planes (2)



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Figure: Values of symmetry are: 0.97, 0.82

Illustration of symmetry planes (3)



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Figure: Values of symmetry are: 0.70, 0.39

Illustration of symmetry planes (4)

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A random set of individuals



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Figure: 30 diverse creatures arranged horizontally according to their values of symmetry (the most symmetrical on the right).

Symmetry in human design and evolution



Figure: Distribution of symmetry values among 84 creatures (38 designed, 46 evolved).

Evolved creatures



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Figure: Evolved creatures. Constructs with the highest symmetry are usually simple.

Designed creatures



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Figure: Designed creatures with symmetry of 1.0.

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- Operates on the phenotype in motion (opposed to: symmetry of genotype)
- Characterizes motion (a feature of the motion pattern).
- Other: whether (to what degree) the movement is periodic or chaotic, how dynamic, effective it is
- Implications:
 - understanding the evolution on Earth
 - methods of locomotion both in living animals and designed robots

Static symmetries





Figure: Symmetry planes of the four considered creatures. Symmetry values are given in brackets.

3D paths

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Figure: Sample 3D paths for four creatures.

2D paths

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Figure: 10 paths for four considered creatures.

Symmetry (3df) over time



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Figure: The values of symmetry over time.

2D paths with symmetries



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Figure: The creature 2D paths (red) with vertical planes shown (green).

Smoothed paths



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Figure: The original paths (red) and the ones smoothed using a low pass filter (blue).

Movement directions





Figure: Movement directions based on the smoothed paths over time.

Vertical (1df) symmetry over time



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Figure: The values of vertical (1df) symmetry over time.

Static symmetries





Figure: Symmetry planes of the four considered creatures. Symmetry values are given in brackets.

Final symmetry values (soft 1df symmetry)

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Table: Soft dynamic 1df symmetries (soft 1df), their standard deviations and maximal and minimal values.

creature	soft 1df	std.dev.	min	max
Basic Quadruped	0.777	0.063	0.588	0.950
Bulldog	0.475	0.062	0.162	0.768
Rototiller	0.688	0.109	0.154	0.932
Imunus Katehe	0.327	0.119	0.090	0.737

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MechaStick

• experiments in 2001: diverse ways of movement evolved

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- experiments in 2001: diverse ways of movement evolved
- were they really diverse?
- mostly simple creatures (a few sticks... large constructs are inefficient)

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- experiments in 2001: diverse ways of movement evolved
- were they really diverse?
- mostly simple creatures (a few sticks... large constructs are inefficient)
- most interesting ones were designed by hand and NNs were evolved

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- new discovery: unexpected numerical instability

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 - high expectations (accuracy, volume bodies, self-collisions)

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 - high expectations (accuracy, volume bodies, self-collisions)
 - evolving movement turned out to be even more difficult! :o
 - elasticity of MechaStick was so important!

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 - sticks as cylinders: rolling ("passive")...and stability phase does not help

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 - sticks as cuboids: instability of simulation, oscillations, and...rolling ("active")

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 - sticks as cuboids: instability of simulation, oscillations, and...rolling ("active")
 - many simulation parameters, each of them is important
 - interdependence between mass, gravity, collision parameters, muscle strength and speed

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 - sticks as cuboids: instability of simulation, oscillations, and...rolling ("active")
 - many simulation parameters, each of them is important
 - interdependence between mass, gravity, collision parameters, muscle strength and speed
 - rolling is a local optimum (so far) demo
- lots of lessons learned...and weeks of simulation.

Further research

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- For which objectives (speed and locomotion, predation, height, etc.) evolution promotes symmetrical creatures?
- Is symmetry beneficial for creatures evolved spontaneously?
- Does symmetry emerge for creatures evolved spontaneously? (evolve, observe, surprise!)
- Which genetical encodings promote symmetry?
- Symmetry as a component of fitness formula.
- Encoding that preserves symmetry. Comparison with other encodings.

References I

[JK08]

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